A Scaling Limit for t-Schur Measures

Sho Matsumoto

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Abstract

We introduce a new measure on partitions. We assign to each partition λ a probability $S_\lambda(x;t)s_\lambda(y)/Z_t$ where s_λ is the Schur function, $S_\lambda(x;t)$ is a generalization of the Schur function defined in [M] and Z_t is a normalization constant. This measure, which we call the t-Schur measure, is a generalization of the Schur measure [O] and is closely related to the shifted Schur measure studied by Tracy and Widom [TW3] for a combinatorial viewpoint.

We prove that a limit distribution of the length of the first row of a partition with respect to t-Schur measures is given by the Tracy-Widom distribution, i.e., the limit distribution of the largest eigenvalue suitably centered and normalized in GUE.

1 Introduction

Let $\mathcal{P}$ be the set of all partitions λ and s_λ the Schur function (see [M]) with variables $x = (x_1, x_2, \dots)$ or $y = (y_1, y_2, \dots)$. The Schur measure introduced in [O] is a probability measure on $\mathcal{P}$ defined by

$$\mathrm{P}_{\mathrm{Schur}}(\{\lambda\}) := \frac{1}{Z_0} s_\lambda(x) s_\lambda(y), \tag{1.1}$$

where the normalization constant Z_0 is determined by the Cauchy identity

$$Z_0 := \sum_{\lambda \in \mathcal{P}} s_\lambda(x) s_\lambda(y) = \prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j}. \tag{1.2}$$

We consider a certain specialization of this measure. Let α be a real number such that $0 < \alpha < 1$. We put $x_i = \alpha$ and $y_j = \alpha$ for $1 \le i \le m$ and $1 \le j \le n$, and let the rest be zero. Fix $\tau = m/n$. This is called the *α-specialization*.

Johansson [J1] (see also [J2], [J3]) showed that when $n \to \infty$ a distribution of the length of the first row λ_1 of a partition λ with respect to the α-specialized Schur measure converges to the Tracy-Widom distribution [TW1], which is the limit distribution of the largest eigenvalue suitably centered and normalized in the Gaussian Unitary Ensemble (GUE).

The Tracy-Widom distribution $F_2(s)$ is explicitly expressed as

$$F_2(s) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[s,\infty)^k} \det(K_{\mathrm{Airy}}(x_i, x_j))_{i,j=1}^k \mathrm{d}x_1 \dots \mathrm{d}x_k,$$

where $K_{\mathrm{Airy}}(x, y)$ denotes the Airy kernel given by

$$K_{\mathrm{Airy}}(x, y) = \int_0^{\infty} \mathrm{Ai}(x + z)\mathrm{Ai}(z + y)\mathrm{d}z$$

and $\mathrm{Ai}(x)$ denotes the Airy function given by

$$\mathrm{Ai}(x) = \frac{1}{2\pi\sqrt{-1}} \int_{\infty e^{-\pi\sqrt{-1}/3}}^{\infty e^{\pi\sqrt{-1}/3}} e^{z^3/3 - xz}\mathrm{d}z.$$

On the other hand, Tracy and Widom [TW3] studied an analogue of the Schur measure, which they call the shifted Schur measure, and proved that a limit distribution of λ_1 with respect to the α-specialized shifted Schur measure is also given by the Tracy-Widom distribution.

In this paper, we introduce a generalization of the Schur measure. In order to define such a new measure, let us recall the symmetric functions $e_n(x;t)$ with parameter t, given by the generating function

$$E_{x,t}(z) := \prod_{i=1}^{\infty} \frac{1 + x_i z}{1 + t x_i z} = \sum_{n=0}^{\infty} e_n(x;t) z^n. \tag{1.3}$$

Then the (generalized) Schur function is given by

$$S_\lambda(x;t) := \det(e_{\lambda'_i - i + j}(x;t)), \tag{1.4}$$

where the partition λ' is the conjugate of a partition λ, i.e., λ'_i is the length of the i-th column of λ. These functions satisfy the so-called Cauchy identity (see [M])

$$Z_t := \sum_{\lambda \in \mathcal{P}} S_\lambda(x;t) s_\lambda(y) = \prod_{i,j=1}^{\infty} \frac{1 - t x_i y_j}{1 - x_i y_j}. \tag{1.5}$$

In particular, when $t = 0$, $E_x(z) = E_{x,0}(z)$ is the generating function of elementary symmetric functions $e_n(x) = e_n(x;0) = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \dots x_{i_n}$. Since $s_\lambda = \det(e_{\lambda'_i - i + j})$ (the dual version of the Jacobi-Trudi identity), we notice that the identity (1.5) becomes (1.2) when $t = 0$.

By means of the identity (1.5), we may define a probability measure on partitions λ by

$$\mathrm{P}_t(\{\lambda\}) := \frac{1}{Z_t} S_\lambda(x;t) s_\lambda(y). \tag{1.6}$$

We call this measure a *t-Schur measure*. It reduces to the Schur measure at $t = 0$.

Our main result is as follows. Denote by P_σ the α-specialized t-Schur measure, where $\sigma = (m, n, \alpha, t)$ is the associated set of parameters of the measure.

Theorem 1. *Suppose $-\infty < t \leq 0$. Then there exist positive constants $c_1 = c_1(\alpha, \tau, t)$ and $c_2 = c_2(\alpha, \tau, t)$ such that*

$$\lim_{n\to\infty} \mathrm{P}_\sigma \left(\frac{\lambda_1 - c_1 n}{c_2 n^{1/3}} < s \right) = F_2(s).$$

This theorem shows that the fluctuations in λ_1 are independent of the parameter t. The assumption that t is non-positive is required since the right hand side of (1.6) must be non-negative after making the α-specialization.

In the case where $t = 0$, this theorem gives the result due to Johansson [J1]. In Remark 2, we shall give the explicit expressions of $c_1(\alpha, \tau, 0)$ and $c_2(\alpha, \tau, 0)$ and explain the connection to the result in [J1]. Note also that Theorem 1 does not imply the result of [TW3] (see Remark 1). The proof of Theorem 1 will be given using the method of Tracy-Widom [TW3]. The key of the proof is the determinantal expression of $S_\lambda(x; t)$.

Further we give a combinatorial interpretation of the t-Schur measure. Namely, if we denote by $\mathbb{P}$ an ordered set $\{1' < 1 < 2' < 2 < 3' < 3 < \dots\}$, then by virtue of the Robinson-Schensted-Knuth (RSK) correspondence between matrices with entries in $\mathbb{P} \cup \{0\}$ and pairs of tableaux of the same shape $\lambda = (\lambda_1, \lambda_2, \dots)$, we see that the t-Schur measure corresponds to a measure (depending on t) on $\mathbb{P}$-matrices. According to this correspondence, λ_1 corresponds to the length of the longest increasing subsequence in the biword w_A associated with a $\mathbb{P}$-matrix A. Using the RSK correspondence and the shifted RSK correspondence, we find that this measure on $\mathbb{P}$-matrices at $t = 0$ and $t = -1$, respectively, corresponds to the (original) Schur measure and the shifted Schur measure, respectively (see [J1] and [TW3]).

2 Schur functions, marked tableaux and the RSK correspondence

In this section, we summarize basic properties of Schur functions and marked tableaux for providing a combinatorial interpretation of the t-Schur measure (see [M] and [Sa2] for details).

We denote the Young diagram of a partition λ by the same symbol λ. Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{P}$ the totally ordered alphabet $\{1' < 1 < 2' < 2 < \dots\}$. The symbols $1', 2', 3', \dots$ or $1, 2, 3, \dots$ are said to be *marked* or *unmarked*, respectively. When it is not necessary to distinguish a marked element k' from the unmarked one k, we write it by $|k|$. A *marked tableau* T of shape λ is an assignment of elements of $\mathbb{P}$ to the squares of the Young diagram λ satisfying the two conditions.

T1 The entries in T are weakly increasing along each row and down each column.

T2 For each $k \geq 1$, each row contains at most one marked k' and each column contains at most one unmarked k.

The condition **T2** says that for each $k \geq 1$ the set of squares labelled by k (resp. k') is a horizontal (resp. vertical) strip.

For example,

$$\begin{array}{llllll} 1' & 1 & 1 & 2' & 3' & 3 \\ 1' & 2 \\ 3 & 3 \end{array}$$

is a marked tableau of shape (6, 2, 2).

To each marked tableau T, we associate a monomial $x^T = \prod_{i \geq 1} x_i^{m_i(T)}$, where $m_i(T)$ is the number of times that $|i|$ appears in T. In the example above, we have $x^T = x_1^4 x_2^2 x_3^4$.

By the definition of $S_\lambda(x;t)$ and Chapter I, §5, Example 23 in [M], it follows that

$$S_\lambda(x;t) = \sum_T (-t)^{\mathrm{mark}(T)} x^T, \tag{2.1}$$

where the sum runs over all marked tableaux T of shape λ. Here mark(T) is the number of marked entries in T. In particular, we have

$$s_\lambda(x) = S_\lambda(x;0) = \sum_T x^T,$$

where the sum runs over all marked tableaux which have no marked entries (i.e., all semi-standard tableaux) of shape λ.

We next explain the RSK correspondence between $\mathbb{P}$-matrices and pairs of tableaux (see [K], [Sa1] and [HH]). Here $\mathbb{P}$-matrix stands for the matrix whose entries are in $\mathbb{P}_0 = \mathbb{P} \cup \{0\}$. To each $\mathbb{P}$-matrix $A = (a_{ij})$ we associate a *biword* w_A as follows. For $i, j \geq 1$, the pair $\begin{pmatrix} i \\ j \end{pmatrix}$ is repeated $|a_{ij}|$ times in w_A, and if a_{ij} is marked, the lower entry j of the first pair $\begin{pmatrix} i \\ j \end{pmatrix}$ appeared in w_A is marked. For example,

$$A = \begin{pmatrix} 1' & 0 & 2 \\ 2 & 1 & 2' \\ 1' & 1' & 0 \end{pmatrix} \longmapsto w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1' & 3 & 3 & 1 & 1 & 2 & 3' & 3 & 1' & 2' \end{pmatrix}.$$

Observe that for a biword $w_A = \begin{pmatrix} \beta_1 & \beta_2 & \dots & \beta_n \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \end{pmatrix}$ the upper line $(\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{N}^n$ is a weakly increasing sequence. Furthermore if $\beta_k = \beta_{k+1}$, then $\alpha_k < \alpha_{k+1}$, or α_k and α_{k+1} are identical and unmarked.

Now we state a generalized RSK algorithm. Let S be a marked tableau and α an element in $\mathbb{P}$. The procedure called an *insertion* of α into S is described as follows.

I1 Set $R :=$ the first row of S.

I2 If α is unmarked, then

I2a find the smallest element β in R greater than α and replace β by α in R. (This operation is called the BUMP.)

I2b set $\alpha := \beta$ and $R :=$ the next row down.

I3 If α is marked, then

I3a find the smallest element β in R which is greater than or equal to α and replace β by α in R. (This is called the EQBUMP.)

I3b set $\alpha := \beta$ and $R :=$ the next row down.

I4 If α is unmarked and is greater than or equal to the rightmost element in R, or if α is marked and greater than every element of R, then place α at the end of the row R and stop.

Write the result of inserting α into S by $I_\alpha(S)$.

For a given biword $w_A = \begin{pmatrix} \beta_1 & \beta_2 & \dots & \beta_n \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \end{pmatrix}$, we construct a sequence of pairs of a marked tableau and a semi-standard tableau as

$$(S_0, U_0) = (\emptyset, \emptyset),\ (S_1, U_1),\ \dots,\ (S_n, U_n) = (S, U).$$

Assuming that a pair (S_{k-1}, U_{k-1}) of the same shape is given. Then we construct (S_k, U_k) as follows. A marked tableau S_k is $I_{\alpha_k}(S_{k-1})$. A semi-standard tableau U_k is obtained by writing β_k into the new cell of U_k created by inserting α_k to S_{k-1}. We call $S = S_n$ a *insertion tableau* and $U = U_n$ a *recording tableau*.

For example, for a biword $w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1' & 3 & 3 & 1 & 1 & 2 & 3' & 3 & 1' & 2' \end{pmatrix}$, we obtain

$$(S, U) = \left(\begin{matrix} 1' & 1 & 1 & 2' & 3' & 3 \\ 1' & 2 \\ 3 & 3 \end{matrix} \ ,\ \begin{matrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 \\ 3 & 3 \end{matrix} \right).$$

The generalized RSK correspondence is then described as follows.

Theorem 2. *There is a bijection between $\mathbb{P}$-matrices $A = (a_{ij})$ and pairs (S, U) of a marked tableau S and an unmarked tableau U of the same shape such that $\sum_i |a_{ij}| = s_j$ and $\sum_j |a_{ij}| = u_i$. Here we put $s_k = m_k(S)$ and $u_k = m_k(U)$ for $k \geq 1$, and the number of marked entries in A is equal to* $\mathrm{mark}(S)$.

Proof. The proof of this theorem can be done in a similar way to the original RSK correspondence between $\mathbb{N}$-matrices and pairs of semi-standard tableaux, or of the shifted RSK correspondence between $\mathbb{P}$-matrices and pairs of shifted marked tableaux (See [HH], [Sa1], [Sa2]). We omit the detail. $\square$

We call the tableau S (resp. U) to be of type $s = (s_1, s_2, \dots)$ (resp. $u = (u_1, u_2, \dots)$).

An important property of this correspondence is its relationship to the length of the longest increasing subsequence in a biword w_A. The *increasing subsequence* in w_A is a weakly increasing subsequence in the lower line in w_A such that a marked k' appears at most one for each positive integer k. In the example above, $(1'\ 1\ 1\ 2\ 3'\ 3)$ is one of such increasing subsequences in w_A. Let $\ell(w_A)$ denote the length of the longest increasing subsequence in w_A. Then we have the

Theorem 3. *If a $\mathbb{P}$-matrix A is corresponding to the pair of tableaux of shape $\lambda = (\lambda_1, \lambda_2, \dots)$ by the generalized RSK correspondence, then we have $\ell(w_A) = \lambda_1$.*

This theorem follows immediately from the following lemma.

Lemma 1. *If $\pi = \alpha_1 \alpha_2 \dots \alpha_n \in \mathbb{P}^n$ and α_k enters a marked tableau S_{k-1} in the jth column (of the first row), then the longest increasing subsequence in π ending in α_k has length j.*

Proof. We prove the claim by induction on k. The result is trivial for $k = 1$. Suppose that it holds for $k - 1$.

First we need to show the existence of an increasing subsequence of length j ending in α_k. Let β be the element of S_{k-1} in the cell $(1, j-1)$. Then we have $\beta < \alpha_k$, or β and α_k are identical and unmarked, since α_k enters in the jth column. By induction, there is an increasing subsequence σ of length $j - 1$ ending in β. Thus $\sigma\alpha_k$ is the desired subsequence.

Now we have to prove that there is no longer increasing subsequence ending in α_k. Suppose that such a sequence exists and let α_i be the preceding element of α_k in the subsequence. Then it is satisfied that $\alpha_i < \alpha_k$, or α_i and α_k are identical and unmarked. Since the sequence obtained by erasing α_k is a subsequence whose length is greater than or equal to j and whose ending is α_i, by induction, α_i enters in some j'th column such that $j' \geq j$ when α_i is inserted. Thus the element γ in the cell $(1, j)$ of S_i satisfies $\gamma \leq \alpha_i$, so that $\gamma < \alpha_k$, or γ and α_k are identical and unmarked.

But since α_k is the element in the cell $(1, j)$ of S_k and $i < k$, we see that $\gamma > \alpha_k$, or that γ and α_k are identical and marked. It is a contradiction. Therefore the lemma follows. $\square$

For a given $\mathbb{P}$-matrix A, λ_1, where $\lambda = (\lambda_1, \lambda_2, \dots)$ is obtained from A by the generalized RSK algorithm above, gives the length of the longest increasing subsequence in a biword w_A. On the other hands, λ_1 obtained by the shifted RSK algorithm between $\mathbb{P}$-matrices and pairs of shifted marked tableaux gives the length of the longest *ascent pair* for a biword w_A associated with A (see in [TW3], [HH]).

3 Measures on $\mathbb{P}$-matrices

We give a combinatorial aspect of the t-Schur measure using the facts stated in the preceding section. Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be variables satisfying $0 \leq x_i, y_j \leq 1$ for all

i, j, and $t \leq 0$. Let $\mathbb{P}_{m,n}$ denote the set of all $\mathbb{P}$-matrices of size $m \times n$. We abbreviate $\ell(w_A)$ to $\ell(A)$ for a $\mathbb{P}$-matrix A. Define a measure depending on a parameter t as follows. Assume that matrix elements a_{ij} in A are distributed independently with the following distributions associated with parameters $x_i y_j$:

$$\begin{aligned}\mathrm{Prob}_t(a_{ij} = k) &= \frac{1 - x_i y_j}{1 - t x_i y_j}(x_i y_j)^k,\\ \mathrm{Prob}_t(a_{ij} = k') &= \frac{1 - x_i y_j}{1 - t x_i y_j}(-t)(x_i y_j)^k\end{aligned}$$

for $k \geq 1$ and

$$\mathrm{Prob}_t(a_{ij} = 0) = \frac{1 - x_i y_j}{1 - t x_i y_j}.$$

This Prob_t indeed defines a probability measure on $\mathbb{P} \cup \{0\}$. Actually we have

$$\sum_{k=0}^{\infty} \mathrm{Prob}_t(|a_{ij}| = k) = 1$$

and $\mathrm{Prob}_t(a_{ij} = k') \geq 0$ for every k since $t \leq 0$.

Let

$$\mathbb{P}_{m,n,s,u,r} := \{A \in \mathbb{P}_{m,n} | \sum_{1 \leq i \leq m} |a_{ij}| = s_j, \sum_{1 \leq j \leq n} |a_{ij}| = u_i, \mathrm{mark}(A) = r\}$$

for $s \in \mathbb{Z}_{\geq 0}^n$, $u \in \mathbb{Z}_{\geq 0}^m$, and $0 \leq r \leq mn$. Here $\mathrm{mark}(A)$ is the number of marked entries in A. Then we have

$$\mathrm{Prob}_t(\{A\}) = \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left(\frac{1 - x_i y_j}{1 - t x_i y_j} \right) (-t)^r x^s y^u = \frac{1}{Z_t}(-t)^r x^s y^u \tag{3.1}$$

for $A \in \mathbb{P}_{m,n,s,u,r}$.

If $\mathrm{Prob}_{t,m,n}$ denotes the probability measure obtained by putting $x_i = y_j = 0$ for $i > m$ and $j > n$, then it follows from (3.1), Theorem 2, Theorem 3, and (2.1) that

$$\begin{aligned}\mathrm{Prob}_{t,m,n}(\ell \leq h) &= \mathrm{Prob}_t(\{A \in \mathbb{P}_{m,n} \mid \ell(A) \leq h\})\\ &= \sum_{s,u,r} \mathrm{Prob}_t(\{A \in \mathbb{P}_{m,n,s,u,r} \mid \ell(A) \leq h\})\\ &= \sum_{s,u,r} \#\{A \in \mathbb{P}_{m,n,s,u,r} \mid \ell(A) \leq h\} \, \frac{1}{Z_t}(-t)^r x^s y^u\\ &= \sum_{s,u,r} \#\{(S, U) \mid \text{type } s \text{ and } u, \ \mathrm{mark}(S) = r, \ \lambda_1 \leq h\} \, \frac{1}{Z_t}(-t)^r x^s y^u\\ &= \frac{1}{Z_t} \sum_{\substack{\lambda \in \mathcal{P} \\ \lambda_1 \leq h}} S_\lambda(x_1, \ldots, x_m; t) s_\lambda(y_1, \ldots, y_n).\end{aligned}$$

A set $\{(S,U)|\text{ type } s \text{ and } u, \operatorname{mark}(S)=r,\ \lambda_1 \le h\}$ consists of all pairs (S,U) of the same shape λ such that $\lambda_1 \le h$, where S is a marked tableau which is of type s and $\operatorname{mark}(S)=r$, and U is a semi-standard tableau which is of type u.

Observe that the rightmost hand side in the above equality is the value of the t-Schur measure with respect to a set $\{\lambda \in \mathcal{P}|\lambda_1 \le h\}$. Particularly, when $t=0$, this measure on $\mathbb{P}$-matrices turns to be the measure on $\mathbb{N}$-matrices and it corresponds to the (original) Schur measure (see [J1], Johansson's q is equal to our α^2). On the other hand, when $t=-1$, by the shifted RSK correspondence we see that it corresponds to the shifted Schur measure (see [TW3]).

Remark 1. The t-Schur measure at $t=-1$ does not coincide with the shifted Schur measure since the correspondences between $\mathbb{P}$-matrices and partitions are different. In fact, Theorem 1 states that a (centered and normalized) limit distribution of $\ell(w_A)$ is identical with one of the length $L(w_A)$ of the longest ascent pair for w_A.

4 Proof of Theorem 1

In this section, we prove Theorem 1 using the methods developed in [TW3]. We recall some notations. Denote the Toeplitz matrix $T(\phi)=(\phi_{i-j})_{i,j\ge 0}$ and the Hankel matrix $H(\phi)=(\phi_{i+j+1})_{i,j\ge 0}$, where $(\phi_n)_{n\in\mathbb{Z}}$ is the sequence of Fourier coefficients of a function ϕ (see [BS]). These matrices act on the Hilbert space $\ell^2(\mathbb{Z}_+)$ ($\mathbb{Z}_+=\mathbb{N}\cup 0$). Also we put $T_h(\phi)=(\phi_{i-j})_{0\le i,j\le h-1}$ and $\tilde{\phi}(z):=\phi(z^{-1})$. Let P_h be the projection operator from $\ell^2(\mathbb{Z}_+)$ onto the subspace $\ell^2(\{0,1,\dots,h-1\})$ and set $Q_h:=I-P_h$, where I is the identity operator on $\ell^2(\mathbb{Z}_+)$.

The following lemmas are keys in the proof. The first one, Lemma 2 is a generalization of the Gessel identity (see [G], [TW2]).

Lemma 2.

$$\sum_{\substack{\lambda\in\mathcal{P}\\ \lambda_1\le h}} S_\lambda(x;t)s_\lambda(y)=\det T_h(\tilde{E}_{x,t}E_y) \tag{4.1}$$

where $E_{x,t}$ is defined in (1.3) *and $E_y=E_{y,0}$.*

Proof. Let $M(x;t)$ be the $\infty\times h$ submatrix $(e_{i-j}(x;t))_{i\ge 1,1\le j\le h}$ of the Toeplitz matrix $T(E_{x,t})$, and for any subset $S\subset\mathbb{N}$, let $M_S(x;t)$ be the submatrix of $M(x;t)$ obtained from rows indexed by elements of S. In particular, write $M(x)=M(x;0)$.

For a partition $\lambda=(\lambda_1,\lambda_2,\dots)$ such that $\lambda_1\le h$, let λ' (whose length is smaller than or equal to h) be the conjugate partition of λ and let $S=\{\lambda'_{h+1-i}+i|1\le i\le h\}$. Then we have $\det M_S(x;t)=\det(e_{\lambda'_{h+1-i}+i-j}(x;t))_{1\le i,j\le h}$. Reversing the order of rows and columns in

this determinant, we have $\det M_S(x;t) = \det(e_{\lambda'_i-i+j}(x;t)) = S_\lambda(x;t)$ by (1.4). In particular, $\det M_S(y) = \det(e_{\lambda'_i-i+j}(y)) = s_\lambda(y)$. It follows that

$$\sum_{\substack{\lambda\in\mathcal{P}\\ \lambda_1\le h}} S_\lambda(x;t)s_\lambda(y) = \sum_S \det M_S(x;t)\det M_S(y)$$

where the sum is all over $S\subset\mathbb{N}$ such that $\#S = h$. Then by the Cauchy-Binet identity, we have

$$= \det\ {}^tM(x;t)M(y) = \det T_h(\tilde{E}_{x,t}E_y).$$

Hence the lemma follows. □

Lemma 3. *If we put $\phi = \tilde{E}_{x,t}E_y$, then we have*

$$\det T_h(\phi) = E(\phi)\det(I - H_1H_2)\mid_{\ell^2(\{h,h+1,\dots\})} \tag{4.2}$$

where put $H_1 = H(\tilde{E}_{x,t}E_y^{-1})$, $H_2 = H(E_{x,t}^{-1}\tilde{E}_y)$ and $E(\phi) := \exp\{\sum_{k=1}^\infty k(\log\phi)_k(\log\phi)_{-k}\}$. Here the determinant on the right side in (4.2) is a Fredholm determinant defined by

$$\det(I-K)\mid_{\ell^2(\{h,h+1,\dots\})} := \det(Q_h - Q_hKQ_h)$$

for any trace class operator K.

Proof. We observe that both H_1 and H_2 are Hilbert-Schmidt operators. We obtain the equality (4.2) by applying directly the relation between the Toeplitz determinant and the Fredholm determinant by Borodin and Okounkov [BoO], [BaW] to ϕ. We leave the detail to the reader. □

Note that for $\phi = \tilde{E}_{x,t}E_y$, we have $E(\phi) = Z_t$.

If we denote by J the diagonal matrix whose i-th entry equals $(-1)^i$, then we have $\det(I-H_1H_2) = \det(I - JH_1H_2J)$. In general, we note that $-JH(\phi(z))J = H(\phi(-z))$ for any function $\phi(z)$.

From (1.6), (4.1) and (4.2), we obtain

$$\mathrm{P}(\lambda_1\le h) = \det(Q_h - Q_hJH_1H_2JQ_h). \tag{4.3}$$

We make here α-specialization and scaling. We put $x_i = y_j = \alpha$ $(0<\alpha<1)$ for $1\le i\le m$ and $1\le j\le n$, and $x_i = y_j = 0$ for $i>m$ and $j>n$. Put $\tau = m/n > 0$. We set $i = h + n^{1/3}x$ and $j = h + n^{1/3}y$, where $h = cn + n^{1/3}s$. The positive constant c will be determined later.

It is convenient to replace $\ell^2(\{h,h+1,\dots,\})$ by $\ell^2(\mathbb{Z}_+)$. Let Λ be a shift operator on $\ell^2(\mathbb{Z}_+)$, i.e., $\Lambda\mathbf{e}_j = \mathbf{e}_{j-1}$ for the canonical basis $\{\mathbf{e}_j\}_{j\ge0}$ of $\ell^2(\mathbb{Z}_+)$ and Λ^* the adjoint operator

of Λ. In $Q_hJH_1H_2JQ_h$, we interpret a Q_h appearing on the left as Λ^h and a Q_h appearing on the right as Λ^{*h}. Then the (i,j)-entry of $-Q_hJH_1J$ is

$$\frac{1}{2\pi\sqrt{-1}}\int\left(\frac{z-\alpha}{z-t\alpha}\right)^m\left(\frac{1}{1-\alpha z}\right)^n z^{-cn-n^{1/3}s-i-j}\frac{\mathrm{d}z}{z^2}$$

and the (i,j)-entry of $-JH_2JQ_h$ is

$$\frac{1}{2\pi\sqrt{-1}}\int\left(\frac{1-t\alpha z}{1-\alpha z}\right)^m(1-\alpha z^{-1})^n z^{-cn-n^{1/3}s-i-j}\frac{\mathrm{d}z}{z^2},$$

where the contours of the integrals are both the unit circle. For the second integral, if we change the variable $z\to z^{-1}$, we obtain

$$\frac{1}{2\pi\sqrt{-1}}\int\left(\frac{z-t\alpha}{z-\alpha}\right)^m(1-\alpha z)^n z^{cn+n^{1/3}s+i+j}\mathrm{d}z.$$

Putting

$$\psi(z)=\left(\frac{z-\alpha}{z-t\alpha}\right)^m\left(\frac{1}{1-\alpha z}\right)^n z^{-cn},$$

we find that two integrals are rewritten as

$$\frac{1}{2\pi\sqrt{-1}}\int\psi(z)z^{-n^{1/3}s-i-j}\frac{\mathrm{d}z}{z^2}, \tag{4.4}$$

$$\frac{1}{2\pi\sqrt{-1}}\int\psi(z)^{-1}z^{n^{1/3}s+i+j}\mathrm{d}z. \tag{4.5}$$

To estimate these integrals, we apply the steepest descent method. At first we determine the constant c. Let $\sigma(z)=n^{-1}\log\psi(z)$, so that

$$\sigma'(z)=\frac{\tau\alpha(1-t)}{(z-\alpha)(z-t\alpha)}+\frac{\alpha}{1-\alpha z}-\frac{c}{z}. \tag{4.6}$$

We choose a constant c such that $\sigma(z)$ has the point z satisfying the equality $\sigma'(z)=\sigma''(z)=0$. Then we obtain

$$\frac{\tau(1-t)(z^2-t\alpha^2)}{(z-\alpha)^2(z-t\alpha)^2}-\frac{1}{(1-\alpha z)^2}=0. \tag{4.7}$$

Since we assume $t\le 0$, it is immediate to see that the function on the left hand side in (4.7) is strictly decreasing from $+\infty$ to $-\infty$ on the interval (α,α^{-1}) and it follows that there is a unique point z_0 in (α,α^{-1}), where the left is equal to zero. This is a saddle point and we set

$$c:=\alpha z_0\left(\frac{\tau(1-t)}{(z_0-\alpha)(z_0-t\alpha)}+\frac{1}{1-\alpha z_0}\right) \tag{4.8}$$

from (4.6). The constant c is positive since $\alpha < z_0 < \alpha^{-1}$ and $t \leq 0$.

It is clear that the number counting with the multiplicity of zeros of the function $\sigma'(z)$ are three from (4.6) when $t \neq 0$. In the case where $t < 0$, $\sigma'(z)$ has a zero in $(t\alpha, 0)$ because $\lim_{z\downarrow t\alpha} \sigma'(z) = -\infty$ and $\lim_{z\uparrow 0} \sigma'(z) = +\infty$. On the other hand, in the case where $t = 0$, the number of zeros of the function $\sigma'(z)$ are two. Therefore, since $\sigma'(z)$ has a double zero z_0, we have $\sigma'''(z_0) \neq 0$. Further, since $\lim_{z\downarrow\alpha} \sigma'(z) = +\infty$ and $\lim_{z\uparrow\alpha^{-1}} \sigma'(z) = +\infty$, $\sigma'(z)$ is positive on (α, α^{-1}) except z_0, so that $\sigma'''(z_0)$ is positive.

We call Γ_+ a steepest descent curve for the first integral (4.4) and Γ_- for the second integral (4.5). On Γ_+, the absolute value $|\psi(z)| = \exp \mathrm{Re}\ \sigma(z)$ is maximal at $z = z_0$ and strictly decreasing as we move away from z_0 on the curve. The first one emanates from z_0 at angles $\pm\pi/3$ with branches going to ∞ in two directions. The second one emanates from z_0 at angles $\pm 2\pi/3$ and is getting close to $z = 0$.

Let D be a diagonal matrix whose i-th entry is given by $\psi(z_0)^{-1} z_0^{n^{1/3}s+i}$, and multiply $Q_h J H_1 H_2 J Q_h = (-Q_h J H_1 J)(-J H_2 J Q_h)$ by D from the left and by D^{-1} from the right. Note that the determinant of the right hand side in (4.3) is not affected.

By the discussion in Section 6.4.1 in [TW3], we see that the $([n^{1/3}x], [n^{1/3}y])$-entry of

$$n^{\frac{1}{3}} D Q_h J H_1 H_2 J Q_h D^{-1}$$

converges to $g K_{\mathrm{Airy}}(g(s+x), g(s+y))$ in the trace norm as $n \to \infty$, where we put

$$g := z_0^{-1} \left(\frac{2}{\sigma'''(z_0)} \right)^{1/3} \tag{4.9}$$

and $K_{\mathrm{Airy}}(x, y)$ is the Airy kernel. Clearly, g is positive.

Hence, by (4.3), we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathrm{P}_\sigma(\lambda_1 \leq cn + n^{1/3}s) \\
&= \sum_{k=0}^{\infty} (-1)^k \sum_{0 \leq l_1 < l_2 < \cdots < l_k} \det\left((D Q_h J H_1 H_2 J Q_h D^{-1})_{l_i, l_j} \right)_{1 \leq i,j \leq k} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[0,\infty)^k} \det\left((D Q_h J H_1 H_2 J Q_h D^{-1})_{[x_i],[x_j]} \right)_{1 \leq i,j \leq k}\ \mathrm{d}x_1 \ldots \mathrm{d}x_k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[0,\infty)^k} \det\left(n^{1/3} (D Q_h J H_1 H_2 J Q_h D^{-1})_{[n^{1/3}x_i],[n^{1/3}x_j]} \right)_{1 \leq i,j \leq k}\ \mathrm{d}x_1 \ldots \mathrm{d}x_k \\
&\longrightarrow \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[0,\infty)^k} \det\left(g K_{\mathrm{Airy}}(g(s+x_i), g(s+x_j)) \right)_{1 \leq i,j \leq k}\ \mathrm{d}x_1 \ldots \mathrm{d}x_k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[gs,\infty)^k} \det\left(K_{\mathrm{Airy}}(x_i, x_j) \right)_{1 \leq i,j \leq k}\ \mathrm{d}x_1 \ldots \mathrm{d}x_k \\
&= F_2(gs).
\end{aligned}$$

This shows the assertion of the theorem, where the constants $c_1(\alpha,\tau,t)$ and $c_2(\alpha,\tau,t)$ are given by c and g^{-1}, respectively. We complete the proof of the theorem.

Remark 2. In the case where $t=0$, we have $z_0 = \frac{\alpha+\sqrt{\tau}}{1+\sqrt{\tau}\alpha}$ by (4.7). Therefore we obtain

$$c_1(\alpha,\tau,0) = \frac{(1+\sqrt{\tau}\alpha)^2}{1-\alpha^2} - 1$$

by (4.8) and

$$c_2(\alpha,\tau,0) = g^{-1} = \frac{\alpha^{1/3}\tau^{-1/6}}{1-\alpha^2}(\alpha+\sqrt{\tau})^{2/3}(1+\sqrt{\tau}\alpha)^{2/3}$$

by (4.9). These values give the corresponding values in Theorem 1.2 in [J1]. Note that the relation between our α and Johansson's q is given by $q=\alpha^2$.

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Sho Matsumoto
Graduate School of Mathematics, Kyushu University.
Hakozaki Fukuoka 812-8581, Japan.
e-mail : `ma203029@math.kyushu-u.ac.jp`